CRITICISMS AND DISCUSSIONS.

PSEUDO-GEOMETRY.

There have appeared in *The Monist,* from time to time, articles treating of "Hyper-space," "Non-Euclidean Geometry," "Pan-Geometry," or what might more appropriately be called "Pseudo-Geometry." I am the more surprised to see these articles go unchallenged in a periodical aiming so scrupulously at rigor in reasoning as does *The Monist.*

"Pan-Geometry" may be very well as a sort of mental gymnastics to test the power of the reasoning faculty by giving it a false or impossible foundation. Such are the old arithmetical puzzles as—"If 2 were 3, what would the half of twenty be?" Likewise the algebraic imaginary; in rigor it is an impossible task since it arises from impossible conditions. The square root of minus one is impossible, and there is no such quantity in reality, since the square of every rational quantity is plus in sign. Yet from the law of signs in algebra we attempt to interpret this irrational quantity and succeed when we treat it as a mere "operator." We reason legitimately as follows: If the square of a negative is a positive and if the numerical value be unity, there is a complete reversal. Then if we consider the expression $\sqrt{-1}$ as a quantity, multiplier, or operator, we find that by using it twice it effects a reversal. If its use twice as a factor effects a complete reversal, then using it once ought to effect a half reversal. If the quantity be treated as a line, then a complete reversal is a rotation through 180 degrees, and a half reversal is a rotation of the line about one extremity through an angle of 90 degrees. General expressions of the form $a+b\sqrt{-1}$ were discussed early in the 19th century by various mathematicians and developed as "theory of complex numbers," "double algebra," and "quaternions." By use of the "operator" Hamilton took a step beyond all previous mathematicians in applying it to a directed line in space, thus founding a new science of mathematics, Quaternions. Yet in all these discussions the expression $\sqrt{-1}$ is not treated as a real quantity, but as a mere sign of operation.

All writers on "pan-geometry," and "hyper-space" to whom my attention has been called found their system on absurd or impossible postulates, and that they reach absurd results is not surprising. They assume that a straight line returns into itself; that the sum of the angles of a plane triangle is less than two right angles, or that it is greater; that the postulate of parallels is untrue, and that parallels meet at an infinite distance.

All these propositions are without validity in truth. Even the last which we meet with so often in mathematical works is untrue and never ought to be stated as true. The real meaning of the statement is that, when we consider the distance of parallels apart, it may be neglected as insignificant in comparison, when treating of lines infinitely long. As a matter of real truth the parallels are as far apart at the infinite distance as they were at the place of beginning. That is, indeed, true by our definition of parallels.

In the January (1906) number of *The Monist* Cassius J. Keyser discusses these subjects under the title "Mathematical Emancipations," endeavoring to develop and make clear to the popular imagination the idea of "hyper-space" from the theory of a "manifold assemblage." Without discussing the paper or commenting on his ability to imagine figures in four-dimensional space, I wish to point out some of his fundamental errors of assumption, leaving others to say whether any confidence can be placed in reasoning based on such foundation.

He assumes a line as a "manifold of points." This is clearly impossible. Continuous quantity cannot be represented by a manifold and its correspondent, number. The very nature of "manifold" makes it discrete and not continuous. A line cannot be made up of mathematical points, since a point has no magnitude, merely position. Two points, however near together, may have an infinite number of interpolations between them. The length of a point is an absolute zero. This multiplied by any infinite or series of infinites is still zero. We are used to saying that a zero multiplied by an infinite produces a finite; but the zero thus considered is merely an infinitesimal, to be rigorously distinguished from the zero absolute, which represents no quantity. An infinity of points with zero distance between them still has no magnitude. An infinity of points in a line represents a discrete quantity and not a continuum, and may be represented by discrete number 1, 2, 3 etc., where we may make the unit as small as we please, but never an absolute zero.

As soon as we give a point magnitude we start with three-dimen-

sional space, and all our reasoning is in three-dimensional space: our line is length, breadth, and thickness, though the latter two may be infinitesimals; our surfaces are length, breadth, and thickness, though the last may be an infinitesimal. It is thus evident that we cannot pass from one order of space to another by any system of multiplication. Points cannot become a line; nor lines, a surface; nor surfaces, a solid.

It is true that a point in motion describes a line and must describe a line. A line may describe a line, if it follow along itself as a path; otherwise its motion must describe a surface. A surface (a plane and surface of a sphere only) may describe a surface if it follows along itself as a path; otherwise it must describe a solid. A solid in motion may describe an infinite number of solids, but its motion always describes a solid. It may be a solid of revolution, a prism, a tortuous prism or cylindroid etc. It cannot be conceived as describing space other than three-dimensional. We have no valid reason for assuming that there is any higher order of space in the ordinary signification than three-dimensional. Indeed there is the best of reasons for assuming that there is none.

Volumes have been written by psychologists and philosophers to explain the notion of space; and the more these philosophers write, the more they seem to think there is something mysterious about it, and the more they themselves become befogged in their reasoning about it. It is a mere matter of definition; everyone knows what it is and can usually define what he means by the word. One might as well attempt to prove that a straight line is straight, or that a circle is circular. Discussions of all self-evident truths are profitless; except it may be to show how such truths are apprehended, and it is extremely doubtful whether these succeed in the sense of giving mental satisfaction.

But what I wish most to protest against, is the use of fundamental untruths as the basis of rigorous reasoning, and insisting on the truth of the deductions when negatived by consciousness and common reason.

LANSING, MICH. CHAS. H. CHASE.

EDITORIAL COMMENT.

This communication on "Pseudo-Geometry" by Judge Charles H. Chase has a certain justification. It is an expression of common sense against being bulldozed into mysticism by the extravagances of a highly abstruse reasoning, and we endorse his protest so far